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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Satellite Doppler Fixation and International Boundaries [and Discussion]

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Satellite Doppler fixation and international boundaries†

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International boundaries have seldom been completely defined in geodetic terms. The existence of natural resources, which ignore the arbitrary boundaries of man, assume considerable importance when division of those resources becomes a point of issue between potential owners. This is particularly so when the boundary is ill-defined in a geodetic sense.

World-wide satellite reference systems, like natural resources, also have little regard for the internally less precise national or international systems. When the one is used to define the location of the other, great care must be taken to ensure equitable division, for financial gain and loss can be considerable. The definition of position is complicated by the existence of the two ephemerides for the N.N.S.S. satellites and the number of alternative reduction procedures available.

The definition of the position of the Frigg Gas Field in the North Sea is an example of how the United Kingdom and Norway resolved the geodetic problem of reconciling geodetic and Doppler data.

Introduction

Disclaimers are often found on mapping, reading to the effect that the map '...is not to be considered authoritative on boundaries or political status except in relation to United Kingdom territories' – or as appropriate. That brief statement conveniently covers the multitude of problems posed to the cartographic community by the many ill-defined, and often politically disputed, international boundaries in the world.

Political awareness has caused the definition of land boundaries to be put in legal terms, with boundary surveys and sometimes physical demarcation as support to the boundary agreements. Most of the present-day boundaries were defined in years when today's precisions were neither attainable, nor even considered to be necessary. If the boundary surveys were based on a triangulation network, the datum was almost certainly astronomically determined with scant regard to deviation of the vertical and with the lack of precise time for longitude determination.

The joining of national geodetic networks across boundaries and the establishment of supra-national datums began to show up the inadequacies of the boundary definitions when pillars, defined in the boundary agreement as having given latitudes and longitudes, were coordinated in the supra-national datum terms and ended up with considerably different values. Where is the boundary? At the latitude and longitude in the boundary agreement, but in which co-ordinate system? Or at the pillar? The existence of the original pillar would make the decision relatively easy, but what if the pillar is gone, or if the boundary was never properly demarcated?

The comparatively small differences between national and supra-national datums had little real importance until recent times when the exploitation of the Earth's natural resources caused man to look much more critically at the extent of his territorial claims. Man in this context

† The views expressed in this paper are those of the author alone and in no way are to be taken as being the views of the United Kingdom Ministry of Defence.

means both the individual, who is sitting on top of his own oilfield, and the nation that will extract taxes from the individual for the pleasure of owning that oilfield.

Until recent times, nations were content to have a simple 3 miles (*ca.* 4.8 km) offshore limit for their territorial waters. But as soon as the exploration geophysicists and developing offshore exploration technology came together to locate valuable oil and gas fields on the continental shelves, the problems of ownership of that wealth became apparent. Not only are nations interested in the ownership of deposits under the sea, but of course they are interested in their offshore limits for fishing rights as well.

Offshore international boundaries have not been demarcated, and knowledge of position relative to the boundary, when out of sight of the shore, depends entirely on some form of navigation aid, be it a radio navigation system, an inertial system or a Doppler satellite receiver. Determination of position offshore to an accuracy of the order of ± 1 m, independent of any geodetic survey network linking back to land-based networks, must rely entirely on satellite Doppler fixation and on acceptable transformation of the Doppler results to the coordinate reference system agreed for the area of operation.

The median line sea boundary

The definition of international sea boundaries within continental shelf areas is a well documented and established process and it calls for the initial definition of national 'baselines' along the coasts, following set rules to cover deep indentations, estuarine and other shoreline irregularities. The turning points along these baselines will be coordinated to various degrees of accuracy in terms of a specific internationally agreed reference system. The accuracy of the coordination of the turning points clearly varies with the nature of the coastline and with the accuracy of the surveying methods available to coordinate the points, which are generally taken to be at low water. In inaccessible areas, errors of up to 30 m may exist in the coordination of the low water mark; elsewhere it may be possible to coordinate to within 3 m relative to the national framework. With the national definition of baselines, demarcation of the territorial limits is through the definition of the median line, every point of which is equidistant from the nearest points of the two baselines.

The definition of the median line becomes a matter of minimum negotiation without the pressures of proven wealth waiting to be divided and with the provision of a common reference system, which was the case in the North Sea in 1965. If the wealth is either strongly suspected or known, the definition of the baselines will become more critical, with more survey effort being put into their coordination, and each change in direction on the median line is liable to be argued before agreement. In the 1965 Agreement on the North Sea, the actual median lines were used only as a basis for negotiation and the agreed boundary was taken as a series of positions defined in geographical coordinates on European Datum 1950 and joined by arcs of 'great circles'. The meaning of the term 'great circle' becomes debatable, however, as soon as greater precision is being considered.

Where adjacent countries are not using the same reference system there is the need to agree on the reference system to be adopted, and whether there is dual definition to cover datum, scale and orientation differences between the two systems. These problems are exemplified in areas such as the Malacca Strait and the Persian Gulf where the existing geodetic control rarely fits well together, either from the use of different datums or through lack of a homo-

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geneous adjustment. Where new 'line of sight' geodetic connections can be made, the problem will be simplified through the ability to produce the homogeneous coordinate reference system on which to base the position of the median line.

In terms, therefore, of the capabilities of satellite Doppler, the definition of the baselines, and hence the median lines, is unsophisticated as must necessarily be the case. However, once the median line is agreed between nations, no matter how arbitrary the original precepts, the turning points become precise points against which division of the underlying natural resources has considerable financial implication.

It is at this stage, where the median line is not visible from the shores, that the use of satellite Doppler to establish position becomes critical. The exploration geophysicists will have defined the extent of the gas or oil field by using a variety of positioning equipments, including broadcast ephemeris satellite Doppler. The broadcast ephemeris will have been converted to the required reference system by using a convenient, but not necessarily wholly acceptable, transformation based on the data available to the organization at that time.

The geophysical data for the oil or gas field will thus be referred to the coordinates of a well head or heads on the sea bottom which are related to 'surface' marks on adjacent drilling or other platforms.

International agreement to the positioning of that particular field against the territorial boundary will then come after careful examination of all the available data, with the recomputation of such Doppler data as are available, or the provision of additional precise Doppler fixes by the national agencies.

The agreement of the location of the Frigg gas field

The North Sea median lines were defined in 1965 in terms of European Datum 1950 as being a reference system which already existed in continental Europe and which could be derived for the United Kingdom through the connections from France to the Ordnance Survey Great Britain 1936 Datum.

Discovery of the Frigg gas field, which was found to straddle the median line between Norway and the United Kingdom, meant that its position was required to be known as accurately as possible against the defined median line, so that an equitable division of the wealth of the field could be made between Norway and the United Kingdom.

The seismic work for the field had been positioned variously in terms of the Decca Cromarty Hi-fix (1971) and the Decca Hordaland and Main OE Bergen chains (1974), and eventually in terms of the then N.N.S.S. Broadcast Ephemeris, APL Mk 4.5 Datum. The shift between APL Mk 4.5 and European Datum 1950 was necessary to position the field correctly against the median line. The construction of Concrete Drilling Platform 1 (C.D.P.1) adjacent to both the median line and well head 25/1–3 (to which the seismic work was referenced) afforded a base for a Doppler receiver in an observation campaign which was commissioned by the field's developers, Elf Norge A/S. The campaign, which was carried out in September–October 1975 by Analytical Technology Laboratories Inc. (A.T.L.), as well as determining the APL Mk 4.5 to European Datum 1950 shift, was to provide data to obtain the best possible European Datum (1950) position for a mark on C.D.P.1. The A.T.L. data were to be made available to both the Norwegian and the United Kingdom governments who would agree the value for the C.D.P.1 mark.

The basis for the work was six precise Doppler fixes which had been observed in 1973–4 by 512 Specialist Team Royal Engineers with some forethought by Norway and the United Kingdom. Three stations were in Norway and three in Scotland and the Shetland Islands, and they therefore straddled that part of the North Sea in which the Frigg gas field is situated. European Datum 1950 coordinates were available for all the stations.

A.T.L. deployed Doppler receivers at these six existing precise Doppler fix sites and on the C.D.P.1 mark, and representatives of the Norwegian and United Kingdom governments monitored the observations. A.T.L. was charged with producing 'quick look' results, having uncertainties in the order of 3 m, while the Norwegian and United Kingdom survey agencies would provide the definitive value for the C.D.P.1 mark. In the event, there was a failure to produce satisfactory data at the station in the Shetland Islands owing to the presence of an uncorrectable large systematic error in all of the Doppler measurements.

For the governmental discussion of the definitive position for C.D.P.1, the A.T.L. observations for the six good stations were reduced against the precise ephemeris by the U.S. DMATC, and the governments thus had the following data available to them: (*a*) six precise ephemeris fixes by 512 S.T.R.E. in 1973–4; (*b*) five precise ephemeris fixes by A.T.L. on shore 1975; (*c*) one precise ephemeris fix by A.T.L. on C.D.P.1 1975; (*d*) European Datum 1950 coordinates for the land stations; (*e*) additional precise fixes in Norway, the United Kingdom and continental Europe with their European Datum 1950 coordinates; (*f*) all of the A.T.L. raw data.

The procedures to obtain the best available European Datum 1950 coordinates for the C.D.P.1 mark were derived from consideration of: (*a*) What basic data would be used? (*b*) Which stations would be used to produce the transformation parameters? (*c*) What method of transformation would be used?

It was agreed that the DMATC precise ephemeris results would be the basis of any computation and the question of how many stations for the transformation gave four possible answers: (*a*) all available Doppler fixes: 9 in Norway and continental Europe and 8 in the United Kingdom; (*b*) the 14 stations south of 62° N; (*c*) the 6 stations occupied by A.T.L.; (*d*) the 5 stations for which A.T.L. results were available.

Norway's need for consideration of her sea boundaries north of 62° N suggested adoption of the 17 station solution for both the C.D.P.1 position and the whole of the North Sea area. Without the addition of data from the Faeroe Islands and more data along the Norwegian coast, any adoption of this approach would clearly pre-empt the results of any future work. The same was true, to a certain extent, for the 14 station solution. The use of the 5 or 6 station transformation, however, would minimize the errors induced by scale and rotations in the various triangulation systems involved in the North Sea ring from Norway to the United Kingdom.

Additional support for the 5 or 6 station transformation came from examining the centres of Laplace rotation which came (reasonably) at a point north of Faeroe Islands for the 6 stations, in southern Portugal for the 14 stations and in mid-Atlantic for the 17 station solutions examined. The overall observation time-scale for the 6 stations, some 5 months in 1973–4, as opposed to 3 years for the 17 stations helped to reduce doubts about corrections for polar motion.

The choice between the use of the 512 S.T.R.E. precise fixes, the A.T.L. fixes reduced on the precise ephemeris or a weighted combination of both, resulted in the adoption of the 512 S.T.R.E. precise fixes, as they were to a programme of carefully balanced satellite passes. Another consideration was that only five of the A.T.L. fixes were available and this imbalance would emphasize the apparent discordance which exists between the Norwegian and United Kingdom triangulation systems in the Y coordinate of the Cartesian frame. Thus the six 512 S.T.R.E. precise fixes made in 1973–4 were adopted to develop the transformation.

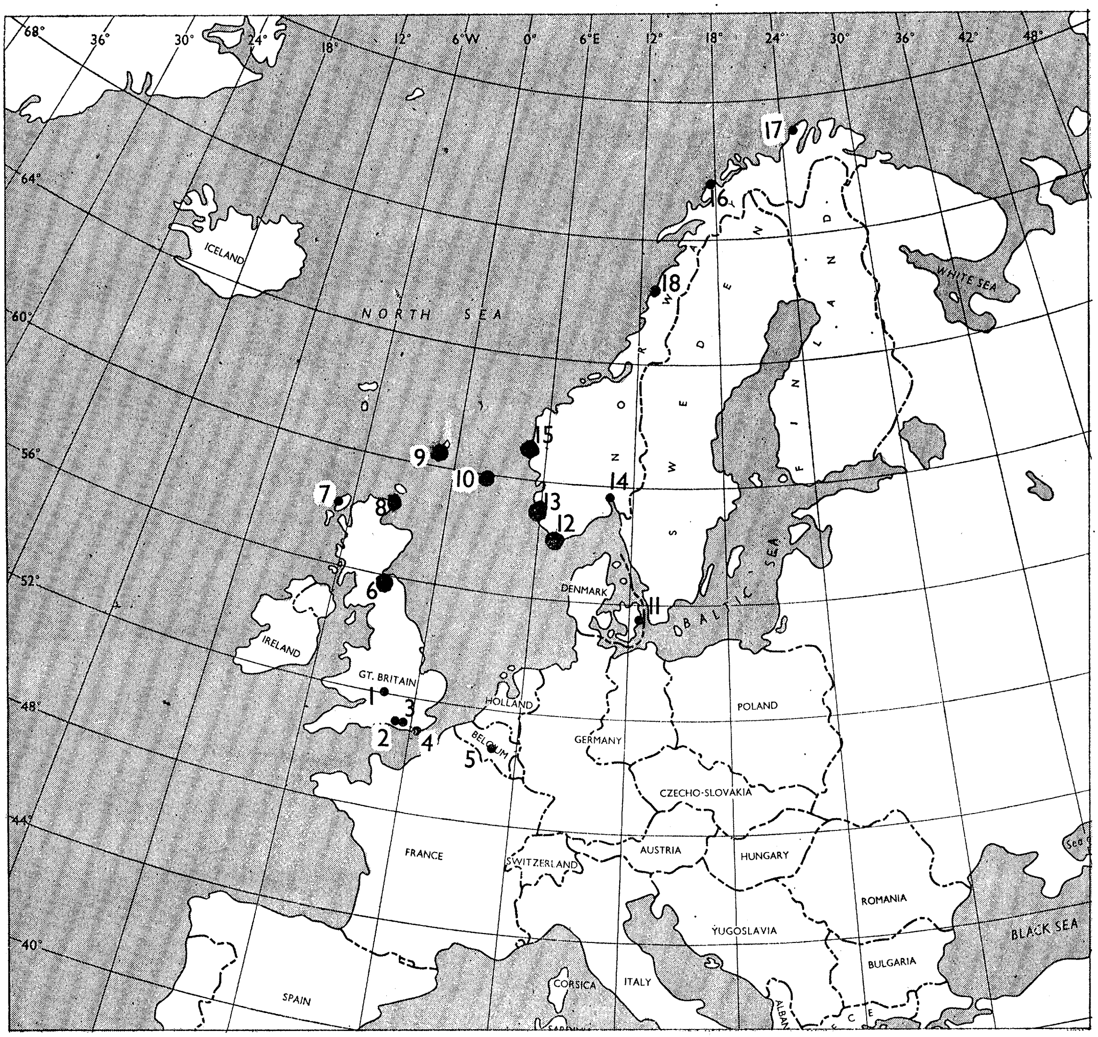

FIGURE 1. Fixation of C.D.P. 1 mark, precise Doppler fix sites: 1, Handgate Farm; 2, Barton Stacey; 3, Lasham; 4, Herstmonceux; 5, Brussels; 6, Earlycoast; 7, Mangersta; 8, Wick; 9, Shetlands; 10, Frigg; 11, Copenhagen; 12, Skibmannshei; 13, Eigeberg; 14, Oslo; 15, Hellisøy Fyr; 16, Tromso; 17, Nord Kapp; 18, Luroy. ●, Stations used in development of the transformation.

The final choice to be made was in the method of transformation, i.e. 3, 4 or 7 parameters. The direct 3 parameter Cartesian shift was adopted for the following principal reasons: (*a*) simplicity; (*b*) C.D.P.1 is comparatively near the controid about which scale and rotation would be applied, and hence the difference introduced by these effects is minimal; (*c*) the residuals at the fixed stations as determined by any of the three methods are well within the expectation of accuracy of the three systems.

Thus the coordinates for the C.D.P.1 mark were obtained from a simple Cartesian shift transformation based on the precise ephemeris results obtained in 1973–4 by 512 S.T.R.E. at

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three stations in Norway and three stations in the United Kingdom and then applied to the precise ephemeris values from A.T.L. observations at C.D.P.1 in 1975 (figure 1).

The agreed value was defined in geographical coordinates on European Datum (1950) as: 59° 52′ 31.70″ N lat.; 02° 03′ 44.75″ E long.; 37.3 m above spheroid; and 29.4 m above mean sea level. The height above mean sea level was derived from a short, simple series of tidal measurements with an estimated accuracy of ± 0.5 m.

Other possible solutions for the C.D.P.1 mark

It is worth considering at this time the differences from the eventually agreed position for the C.D.P.1 mark had any of the other solutions been adopted. Table 1 details the transformed coordinates for C.D.P.1 with the use of the precise ephemeris results for all stations in determining the transformation parameters.

Table 1. C.D.P.1 Doppler station; European Datum 1950

	number of parameters	X/m	Y/m	Z/m
17 stations	3	3207 216.0	115 496.1	5493 685.4
	4	216.1	496.2	685.2
	7	217.0	496.0	685.7
14 stations	3	216.0	497.5	685.2
	4	216.4	497.5	684.9
	7	216.5	496.5	685.1
6 stations	3†	3207 216.6	115 497.5	5493 685.3
	4	216.5	497.6	685.3
	7	216.3	497.0	685.2
	range	1.0	1.6	0.8

† The accepted value.

For the specific purpose of defining the position of the C.D.P.1 mark it is obvious that any of the solutions would have produced an answer acceptable within the likely error of the fix on C.D.P.1.

An estimated accuracy of ± 3 m in each Cartesian axis was quoted for the independent precise ephemeris solution for C.D.P.1, implying a vector accuracy of ± 5 m (approximately) for the Doppler determination for the station. However, it would appear that the relative precision among the six stations used for the transformation was better than this, since the standard error of an observation equation (all given equal weight) was ± 1.8 m, implying a vector accuracy of approximately 3 m in the fit between the two systems; this figure includes the errors of both the Doppler fixations and the terrestrial triangulation networks used for determining the European Datum 1950 coordinates.

The A.T.L. result from computation of the data using their Compunav GP/3D High Precision Satellite Positioning System gave the European Datum 1950 values: $X = 3\,207\,206.7$ m; $Y = 115\,492.0$ m; $Z = 5\,493\,667.1$ m, or 59° 52′ 31.69″ N lat.; 02° 03′ 44.42″ E long. and 25.30 m above mean sea level, which is a little to the west of the accepted value. The reasons for this are not entirely clear, but the difference is mainly in height, which might suggest incorrect spheroid parameters in the solution. The plan solution shows a longitude shift of 0.33″ westwards from the accepted position. The often quoted value for a shift of the median line east or west in the Frigg gas field is $2 $\times 10^6$ per metre.

The agreement of the position of the C.D.P.1 mark by the two governments was the result of a straightforward piece of geodetic manipulation of the data available at that time. It was helped by the fact that anticipatory action to obtain precise Doppler fixes had been taken and that both the Norwegian and United Kingdom control networks could be expressed in terms of European Datum 1950. With the use of the Ordnance Survey Scientific Network 1970 as a basis for the production of the United Kingdom European Datum 1950 coordinates, the scale problems present in the Ordnance Survey Great Britain 1936 adjustment were avoided. Thus the best available relationship of Norway and the United Kingdom across the northern North Sea could be established for the coordination of C.D.P.1.

General considerations of boundary location via satellite Doppler

Procedures similar to those described above can be adopted whenever the participating nations are using a geodetically well defined common reference system, but there are many areas off and on shore where such systems do not exist.

The major considerations are: (*a*) the physical definition of the boundary; (*b*) the relation of the satellite Doppler reference system to local datums; (*c*) the reconciliation of incompatible results from satellite Doppler results and ground surveys; and (*d*) the unique definition of a point fixed on a boundary by satellite Doppler.

The physical definition of the boundary is initially not for the satellite geodesist, but rests either with the lawyers to interpret and agree on existing definitions of the boundary, or with national survey departments to undertake new surveys and, hopefully, to eliminate expressions like 'great circles' from the finally agreed documents. It may well be that the satellite geodesist will eventually become involved in providing control for boundary demarcation, especially in remote areas.

Problems (*b*) and (*c*) above are the main areas of challenge, for even in Europe there are areas where the existing adjustments are weak enough to be shown up by Doppler fixes, and careful procedures are necessary to take into account local variations in the control network, especially if there is to be any extrapolation offshore.

The use of generalized shifts between the satellite reference systems and geodetic networks has limitations which are not always appreciated by all users, and the regional variations can be considerable. For precise work, as in the Frigg gas field positioning, and for the best conversion to the operational reference system, it is better to derive local conversion parameters.

Outside the areas of good geodetic control, the problem of matching ground surveys to satellite reference systems can present considerable difficulties, and one can cite the case of the Persian Gulf as an example. The geodetic history of the southern shores of the Gulf must be without parallel in the rest of the world. The multiplicity of datums and adjustments which have been produced in the general exploration for, and exploitation of, the oil and gas have caused confusion for a long time. Fortunately the accuracies now obtainable from the N.N.S.S., even with the broadcast ephemeris, allow some rational assessment to enable the user to judge at least whether the ground coordinates he is working with are all in a homogeneous system. Lack of definition of national boundaries in the past, both on shore and their seaward extensions, is now being rectified in the Gulf States, but there is still no overall reference system generally available in that part of the world. Conversion from WGS 72 to the local version of Nahrwan Datum is very much a question of detailed local knowledge.

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The provision of a unique coordinate definition for a point from satellite Doppler is unattainable, owing to many factors. The two ephemerides will give different results from the same set of observations when reduced to a common datum, different reduction procedures will give different results, and the nature of the surface surrounding the antenna affects the signal reception and hence the resulting position. There are many others. Admittedly most of these differences should be within the 'noise' of N.N.S.S., but there is still a great deal to be learnt about the refinements still available to the user. One of the currently more worrying aspects about N.N.S.S. is the lack of direct compatibility between the precise and broadcast ephemerides and the resulting need to work in one or the other system, but not to mix them.

The power of the precise ephemeris individual point positioning as an external arbitrator must never be overlooked when boundaries are being considered, because it provides the only independent check on existing ground-based reference systems.

Future considerations

The aim of any definition of a territorial boundary should be to produce an unambiguous statement of the extent of that boundary. We are aware that many existing national boundaries are far from being unambiguous and much time and money is spent, sometimes frustratingly, in search of agreement.

The advent of satellite geodesy has meant that the concept of a meaningful world reference system is with us now, and we have the means to connect every survey on Earth into a world reference system. Nations, in this twentieth century, have little desire or need to convert their existing reference systems into a world system of reference. However, it is not too difficult to envisage that, at some time in the future, the introduction of a world reference system worldwide will be both feasible and desirable.

We have seen the rapid development of world geodetic reference systems through to WGS 72 which anticipates the post-1980 definition. With each successive definition the degrees of uncertainty have been significantly reduced. We have now reached the stage where further developments in the reference system will be of geodetic, rather than topographic, interest.

The emphasis in the future exploitation of the Earth's resources is going to be on the continental shelves, with the consequent offshore positioning requirements. It would seem logical that the definition of any future international boundary and exploration block, both at sea and on land, should be in terms of the current World Geodetic System to provide a nearly unambiguous and recoverable boundary. The subtle differences in reduction programs due to the variations in the methods used will always ensure that an unambiguous value is unattainable.

Clearly the exploration geophysicist, working offshore and in remote land areas, would benefit from boundary definition in the same terms as his referencing system because it would remove the intermediate and often uncertain conversion stage from his calculations. Developments of real-time position navigation systems such as Navstar G.P.S., which will be referenced in WGS 72 (or its successor), add weight to the need for the geodetic community to advise the international lawyers of the need for a review of the terms in which international and resource exploration boundaries are defined.

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Discussion

J. C. Blankenburgh (*Continental Shelf Institute, Trondheim, Norway*). Why did the A.T.L. results not agree with the other results?

N. A. G. Leppard. Apart from the possibility of different spheroid parameters mentioned in the paper, there is the fact that a different approach to the use of the observed data was made by A.T.L.

P. A. Cross (*Department of Land Surveying, North East London Polytechnic, Forest Road, London E*17 4*JB, U.K.*). How accurately can the boundaries of oil or gas fields be determined from geophysical data?

N. A. G. Leppard. I am led to understand that it is possible to define the boundaries to within about 5 m, but there may be someone here who has a better knowledge of the problem.

P. G. Sluiter (*c/o Shell, EP/*12, *P.O. Box* 162, *The Hague, Netherlands*). I am convinced that the actual location of subsurface mineral resources is known to a much smaller accuracy than the 5 m mentioned by Mr Leppard. I would suggest a value of 25 m, though it can be much worse. Some reasons for this are the difficulty in interpreting results of seismic surveys, the feathering angle of the cable and reflexions returning from features outside the vertical profile underneath the ship's track. Nor is positioning the only problem in trying to determine how much oil belongs to who. The thickness of the oil-bearing strata and their porosity and hence the actual recoverable quantity are in many cases more important than the actual horizontal position.

N. A. G. Leppard. As geodesists, we should concern ourselves with providing the best possible value for a point to which the geophysicist can relate his data which, as we have just heard, are often to a very much lower order of accuracy.

S. Bakkelid (*Norges geografiske oppmåling, Postboks* 8153, *Dep., Oslo* 1, *Norway*). The two last papers have demonstrated the need for a definition of ED 50 in the North Sea. To that end we have to use the Doppler system, which is the only useable one at present. The transformation constants or formulae for converting Doppler to ED 50 in the North Sea constitute definitions of ED 50 by Doppler in this area. They were derived from a comparison of Doppler and ED 50 coordinates of points mainly in U.K. and Norway available in 1976. Let us, however, take a closer look at what we are up against. In the U.K. and Norway we have in each country a special version of ED 50 with its own scale and orientation. The definition of ED 50 in the intermediate space should be based on this fact.

The first thing to do is therefore to determine transformation formulae separately for the land areas of the U.K. and Norway. The second problem is to determine the transformation formula for the North Sea by combining the two formulae for the land areas in such a way that a consistent ED 50 system is obtained for the total area and a smooth transition of ED 50 from the Norwegian coast over the North Sea on to the coast of U.K. This can be done in different ways. One is to weight the two original formulae in proportion to the inverse distance to the nearest points on the two baselines. This leads to a transformation formula that is a linear combination of the two basic formulae. In the Geographical Survey of Norway we have determined a transformation formula for the North Sea with the use of this method and the Doppler and ED 50 data available in 1976. We have also determined by co-location transformation formulae based on the additional stipulation that the points on the median lines in the North Sea, determined in ED 50 as equidistant points relative to the baseline points, should retain this characteristic when converted to the Doppler system. This requirement was met by the first method only within about 1.5 m.

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